Week 11 - The Bott-Samelson Theorem

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1 Instructions

This week we will prove the Bott-Samelson Theorem and compute the cohomology of ΩS^{n+1}. All the exercises appear in sections 6 and 7. Please complete all of them.

2 The Coalgebra Structure on Homology

Let X be a space and assume that either H_*X is torsion free or that coefficients are taken in a field. In this case the homology cross product is an isomorphism

$$-\times - : H_*X\otimes H_*X\xrightarrow{\cong} H_*(X\times X). \tag{2.1}$$

This is the content of the *Künneth Theorem*. We denote by $\Delta=\Delta_X : X\to X\times X$ the diagonal map of X and consider the composite

$$\Delta_* : H_*X\to H_*(X\times X)\cong H_*X\otimes H_*X. \tag{2.2}$$

This map endows H_*X with the structure of a **coalgebra**, so called because the projections onto the two factors give rise to a commutative diagram

$$\begin{tikzcd} & H_*X \arrow[dl, "\cong"'] \arrow[d, "\Delta_*"] \arrow[dr, "\cong"] & \\ H_*X & H_*X \otimes H_*X \arrow[l, "pr_{1*}"] \arrow[r, "pr_{2*}"'] & H_*X. \end{tikzcd} \tag{2.3}$$

If you want to understand where the name comes from simply turn the arrows of this diagram around to find something familiar. The diagram shows that if $u \in H_*X$, then

$$\Delta_* u = u \otimes 1 + \textstyle\sum_i u_i' \otimes u_i'' + 1 \otimes u \tag{2.4}$$

where the $u_i', u_i'' \in H_*X$ are classes with degrees satisfying $|u_i'| + |u_i''| = |u|$. Of course the u_i', u_i'' may be zero.

If the idea of the coproduct seems abstract recall that the cup product in cohomology is induced by the diagonal map. That is, if $x, y \in H^*X$, then

$$x \cup y = \Delta^*(x \otimes y) \tag{2.5}$$

The point is that (2.4) contains exactly the same information as the cup product. In fact the coproduct has more obvious geometric significance.

To relate the coproduct on H_*X with the cup product on H^*X we have the Kronecker duality pairing

$$\langle -, - \rangle : H^*X \otimes H_*X \to R. \tag{2.6}$$

This gives us

$$\langle x \cup y, u \rangle = \langle \Delta^*(x \otimes y), u \rangle = \langle x \otimes y, \Delta_* u \rangle \tag{2.7}$$

and if $\Delta_* u$ is as in (2.4), then this expands out as

$$\begin{aligned} \langle x \otimes y, \Delta_* u \rangle &= \langle x \otimes y, u \otimes 1 \rangle + \textstyle\sum_i \langle x \otimes y, u_i' \otimes u_i'' \rangle + \langle x \otimes y, 1 \otimes u \rangle \\ &= \langle x, u \rangle \cdot \langle y, 1 \rangle + \textstyle\sum_i \langle x, u_i' \rangle \cdot \langle y, u_i'' \rangle + \langle x, 1 \rangle \cdot \langle y, u \rangle. \end{aligned} \tag{2.8}$$

Thus u is dual to $x \cup y$ if and only if x, y are dual to terms appearing in (2.4).

Example 2.1 *Suppose X is a space with $H_k X = 0$ for $k < n$. Then if $u \in H_n X$ we have*

$$\Delta(u) = u \otimes 1 + 1 \otimes u \tag{2.9}$$

for dimensional reasons. In fact, the same equation must hold whenever $u \in H_k X$ for $n \leq k < 2n$. This simple observation characterises the coproduct in $H_* S^n$ completely. $\square$

Example 2.2 *We compute the coproduct in $H_* \mathbb{C}P^n$. Let $x \in H^2 \mathbb{C}P^2$ be a generator. Then we define $u_1 \in H_2 \mathbb{C}P^2$ and $u_2 \in H^4 \mathbb{C}P^2$ to be the unique classes satisfying*

$$\langle x, u_1 \rangle = 1, \qquad \langle x^2, u_2 \rangle = 1 \tag{2.10}$$

As in the previous example we have

$$\Delta_* u_1 = u_1 \otimes 1 + 1 \otimes u_1. \tag{2.11}$$

To compute $\Delta_* u_2$ we check that

$$\langle x \otimes x, \Delta_* u_2 \rangle = \langle \Delta^*(x \otimes x), u_2 \rangle = \langle x^2, u_2 \rangle = 1 \tag{2.12}$$

and similarly $\langle x^2 \otimes 1, \Delta_* u_2 \rangle = 1 = \langle 1 \otimes x^2, \Delta_* u_2 \rangle$, and so conclude that

$$\Delta_* u_2 = u_2 \otimes 1 + u_1 \otimes u_1 + 1 \otimes u_2. \tag{2.13}$$

With a little more work we can figure out the diagonal in $\mathbb{C}P^n$. Let $u_i \in H_{2i}\mathbb{C}P^n$ be dual to $x^i \in H^{2i}\mathbb{C}P^n$. Then arguing as above we find

$$\Delta u_i = u_i \otimes 1 + \sum_{j=1,\dots i-1} u_{i-j} \otimes u_j + 1 \otimes u_i. \tag{2.14}$$

□

3 Pontryagin Algebras

Assume that (X, μ) is an H-space. Then the composition

$$H_*X \otimes H_*X \xrightarrow{\times} H_*(X \times X) \xrightarrow{\mu_*} H_*X \tag{3.1}$$

turns H_*X into a graded algebra. The first map here is the homology cross product, which is always injective. We call (3.1) the **Pontryagin product**, and refer to H_*X with this algebra structure as the **Pontryagin Algebra** of (X, μ). The Pontryagin product has a unit. It is associative (commutative) whenever (X, μ) is a homotopy associative (commutative), and it has inverses whenever (X, μ) has an H-inverse.

The coproduct on H_*X pairs well with the Pontryagin product in the following sense. The space $X \times X$ is an H-space with multiplication

$$(X \times X) \times (X \times X) \xrightarrow{1 \times T \times 1} X \times X \times X \times X \xrightarrow{\mu \times \mu} X \times X \tag{3.2}$$

where T is the twist map. This means that $H_*(X \times X)$ has its own Pontryagin product. Still working under the assumption that H_*X is torsion free, or that we have coefficients in a field, we check that the following diagram commutes

$$\begin{tikzcd} H_*X \otimes H_*X \arrow[rrr, "\mu_*"] \arrow[d, "\Delta_* \otimes \Delta_*"'] & & & H_*X \arrow[d, "\Delta_*"] \\ H_*X \otimes H_*X \otimes H_*X \otimes H_*X \arrow[rr, "1 \otimes T \otimes 1"] & & H_*X \otimes H_*X \otimes H_*X \otimes H_*X \arrow[r, "\mu_* \otimes \mu_*"] & H_*X \otimes H_*X. \end{tikzcd} \tag{3.3}$$

This diagram expresses the fact that Δ_* is an algebra homomorphism.

Proposition 3.1 *Under the stated assumptions, the diagonal on X induces an algebra homomorphism*

$$H_*X \to H_*(X \times X) \cong H_*X \otimes H_*X. \tag{3.4}$$

where the domain and codomain carry their Pontryagin structures. ▮

For an example of the utility of the proposition

Example 3.1 *Give S^3 its standard Lie structure inherited from the quaternions. Then $\mathbb{R}P^3 = S^3/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ subgroup is generated by ± 1. Since this subgroup is central there is an induced multiplication on the quotient $\mathbb{R}P^3$. In fact with a little work we can show that $\mathbb{R}P^3$ with this multiplication is isomorphic to the Lie group SO_3 of rotations of 3-dimensional space. Thus we can calculate the Pontryagin algebra of $\mathbb{R}P^3 \cong SO_3$, at least with $\mathbb{Z}_2$ coefficients.*

We take all homology with coefficients in $\mathbb{Z}_2$. Then for $i = 1, \ldots, 3$ we have $H_i\mathbb{R}P^3 \cong \mathbb{Z}_2$, say generated by u_i. We would like to determine the Pontryagin products $u_1 \cdot u_1$ and $u_1 \cdot u_2$. Now a calculation similar to that of Example (2.1) *shows that*

$$\Delta_* u_i = u_i \otimes 1 + \sum_{j=1,\ldots i-1} u_{i-j} \otimes u_j + 1 \otimes u_i \tag{3.5}$$

for each $i = 1, 2, 3$. With $\mathbb{Z}_2$-coefficients we find

$$\Delta_*(u_1 \cdot u_1) = \Delta_*(u_1) \cdot \Delta_*(u_1) = (u_1 \otimes 1 + 1 \otimes u_1)^2 = u_1^2 \otimes 1 + 1 \otimes u_1^2 \tag{3.6}$$

and since this is not equal to $\Delta_* u_2$ it must be that $u_1 \cdot u_1 = 0$. On the other hand we have

$$\Delta_*(u_1 \cdot u_2) = \Delta_*(u_1) \cdot \Delta_*(u_2) = (u_1 \cdot u_2) \otimes 1 + u_1 \otimes u_2 + u_2 \otimes u_1 + 1 \otimes (u_1 \cdot u_2). \tag{3.7}$$

Now this implies that $u_1 \cdot u_2 \neq 0$, since if it were, then both sides of this equation would vanish. The middle two terms on the right-hand side show that this cannot happen.

Above we have shown that $u_1 \cdot u_1 = 0$ and $u_1 \cdot u_2 = u_2 \cdot u_1 = u_3$. It follows that the Pontryagin algebra

$$H_*\mathbb{R}P^3 \cong \Lambda(u_1, u_2) \tag{3.8}$$

is an exterior algebra.

Similarly $\mathbb{R}P^7$ inherits a (non-associative) multiplication from S^7. Running similar computations to the above we can show that

$$H_*\mathbb{R}P^7 \cong \Lambda(u_1, u_2, u_4). \tag{3.9}$$

□

3.1 An Example

The Lie group SU_n can be described as the set of all complex $n \times n$ matrices A which satisfy

1. $A^\dagger A = I$, where the dagger denotes Hermitian transpose.

2. $\det(A) = 1$.

Then SU_n acts on $\mathbb{C}^n$ in a way which fixes the unit sphere S^{2n-1} and we can define an evaluation map

$$p : SU_n \to S^{2n+1}, \qquad A \mapsto A \cdot e_1. \tag{3.10}$$

It can be shown that p is a locally trivial fibration and we check that $p^{-1}(e_1) = SU_{n-1}$. Thus there is a fibration sequence

$$SU_{n-1} \to SU_n \to S^{2n-1} \tag{3.11}$$

which by an inductive process gives us access to the low-dimensional homotopy groups of SU_n.

The group SU_1 is a single point. Checking directly we see that SU_2 consists of complex matices of the form

$$A = \begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix} \tag{3.12}$$

where $z_1, z_2 \in \mathbb{C}$ satisfy $|z_1|^2 + |z_2|^2 = 1$. In this case the evaluation map $p : SU_2 \to S^3$ is the diffeomorphism $A \mapsto (z_1, z_2)$. Better yet, when we identify S^3 with the unit sphere in $\mathbb{H}$ using the map $(z_1, z_2) \mapsto z_1 + jz_2$ we see that p is in fact an isomorphism of Lie groups (recall that the Lie product on S^3 was defined as that inherited from the quaternionic multiplication).

Now when $n = 3$ the isomorphism $SU_2 \cong S^3$ gives us a fibration sequence

$$S^3 \to SU_3 \to S^5. \tag{3.13}$$

Following the clutching construction which was described in *Fibrations III* we see that SU_3 may be obtained up to homeomorphism as a pushout

$$\begin{tikzcd} S^4 \times S^3 \arrow[r, hook] \arrow[d, "\varphi"'] & D^5 \times S^3 \arrow[d] \\ D^5 \times S^3 \arrow[r] \arrow[ur, phantom, "\lrcorner"] & SU_3 \end{tikzcd} \tag{3.14}$$

for some map φ. The pushout is also a homotopy pushout and studying its Mayer-Vietoris sequence we get

$$H_k SU_3 \cong \begin{cases} \mathbb{Z} & k = 0, 3, 5, 8 \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

Since the homology is torsion free we have $H^*SU_3 \cong Hom(H_*SU_3, \mathbb{Z})$, so (3.15) also describes the cohomology groups of SU_3. For $i = 3, 5, 8$ let $u_i \in H_i SU_3$ be generators and $x_i \in H^i SU_3$ their duals, so that $\langle x_i, u_j \rangle = \delta_{ij}$.

Now, with the Pontryagin product $u_3 \cdot u_5 \in H_8 SU_3$, so this element is some (possibly zero) multiple of u_8. To find out more we use Proposition 3.1 to get

$$\Delta_*(u_3 \cdot u_5) = \Delta_*(u_3) \cdot \Delta_*(u_5). \tag{3.16}$$

Of course $\Delta_*(u_3) = u_3 \otimes 1 + 1 \otimes u_3$ and $\Delta_*(u_5) = u_5 \otimes 1 + 1 \otimes u_5$ for dimensional reasons, so (3.16) expands as

$$\begin{aligned} \Delta_*(u_3 \cdot u_5) &= (u_3 \otimes 1 + 1 \otimes u_3) \cdot (u_5 \otimes 1 + 1 \otimes u_5) \\ &= (u_3 \cdot u_5) \otimes 1 + u_3 \otimes u_5 - u_5 \otimes u_3 + 1 \otimes (u_3 \cdot u_5). \end{aligned} \tag{3.17}$$

Note the -1 sign that appears from the graded commutation rule $(1 \otimes u_5) \cdot (u_3 \otimes 1) = (-1)^{|u_3||u_5|} u_3 \otimes u_5$.

On the other hand, in cohomology we have that $x_3 \cup x_5$ is some (possibly zero) multiple of x_8. To see that this cup product is non-trivial we compute

$$\begin{aligned}\langle x_3 \cup x_5, u_3 \cdot u_5 \rangle &= \langle \Delta^*(x_3 \otimes x_5), u_3 \cdot u_5 \rangle = \langle x_3 \otimes x_5, \Delta_*(u_3 \cdot u_5) \rangle \\ &= \langle x_3 \otimes x_5, (u_3 \cdot u_5) \otimes 1 \rangle + \langle x_3 \otimes x_5, u_3 \otimes u_5 \rangle + \dots \\ &= 0 + \langle x_3, u_3 \rangle \langle x_5, u_5 \rangle + 0 + \dots \\ &= 1\end{aligned} \tag{3.18}$$

with all other terms on the second line evaluating to 0. The only way that (3.18) can be non-trivial is if both $x_3 \cup x_5 \neq 0$ and $u_3 \cdot u_5 \neq 0$. The fact that it evaluates to 1 implies that actually $x_3 \cup x_5 = x_8$ and $u_3 \cdot u_5 = u_8$. It follows that

$$H^* SU_3 \cong \Lambda(x_3, x_5) \tag{3.19}$$

(under the cup product) is an exterior algebra, as is

$$H_* SU_3 \cong \Lambda(u_3, u_5) \tag{3.20}$$

(under the Pontryagin product).

The results of this calculation generalise, although the ad hoc methods do not. In general

$$H^* SU_n \cong \Lambda(x_3, x_5, \dots, x_{2n-1}) \tag{3.21}$$

and

$$H_* SU_n \cong \Lambda(u_3, u_5, \dots, u_{2n-1}) \tag{3.22}$$

where x_{2i-1} is dual to u_{2i-1}. A calculation can be found in Steenrod-Epstein [1] Chptr IV, where (3.21) is obtained as a Corollary of the more fundamental computation (3.22).

4 The Suspension Map

Fix a space X. By taking the adjoint of the identity $id_{\Sigma X} : \Sigma X \xrightarrow{=} \Sigma X$ we obtain a map

$$\sigma = \sigma_X : X \to \Omega \Sigma X. \tag{4.1}$$

By definition

$$\sigma(x)(t) = x \wedge t, \qquad x \in X,\ t \in I. \tag{4.2}$$

We call σ the **suspension** map of X. Its appearance is related to the work of the previous sections because we will be interested in the homology of the loop space $\Omega \Sigma X$. The name comes from the fact that if $\Sigma : [A, X] \to [\Sigma A, \Sigma X]$ denotes the assignment $f \mapsto \Sigma f$, then

$$\begin{tikzcd} {[A,X]} \arrow[r, "\Sigma"] \arrow[d, equal] & {[\Sigma A, \Sigma X]} \arrow[d, "\cong"] \\ {[A,X]} \arrow[r, "\sigma_*"] & {[A, \Omega\Sigma X]} \end{tikzcd} \tag{4.3}$$

commutes, where the right-hand bijection is the adjunction. Thus σ is the topological counterpart to the abstract operation Σ.

There are deep things that can be said about the suspension map. For instance the *Freudenthal Suspension Theorem* gives conditions under which σ_* is bijective, although this is not something we'll explore today

The suspension is natural. If $f : X \to Y$ is given, then

$$\begin{tikzcd} X \arrow[r, "\sigma_X"] \arrow[d, "f"'] & \Omega\Sigma X \arrow[d, "\Omega\Sigma f"] \\ Y \arrow[r, "\sigma_Y"] & \Omega\Sigma Y \end{tikzcd} \tag{4.4}$$

is commutative. Moreover, if a map $f : X \to \Omega Y$ into a loop space is given, then f factors through the suspension as

$$f : X \xrightarrow{\sigma_X} \Omega\Sigma X \xrightarrow{\Omega f^\flat} \Omega Y \tag{4.5}$$

where $f^\flat : X \to Y$ is the adjoint of f.

5 A Lemma

It is not immediately clear how this lemma fits into the ideas of previous sections. Nevertheless it will be a crucial part of our proof of the Bott-Samelson Theorem.

Proposition 5.1 *Let $p : E \to B$ be a fibration and $j : A \hookrightarrow B$ a closed cofibration. Then the inclusion $E_A = p^{-1}(A) \hookrightarrow E$ is a closed cofibration.*

Proof Equip the inclusion $j : A \hookrightarrow B$ with a Strøm structure

$$\varphi : B \to I, \qquad G : B \times I \to B \tag{5.1}$$

and let H be a filler in the diagram

$$\begin{tikzcd} E \arrow[rr, "id"] \arrow[d, "in_0"'] & & E \arrow[d, "p"] \\ E \times I \arrow[r, "p\times 1"'] \arrow[urr, dashed, "H"] & B \times 1 \arrow[r, "G"'] & B. \end{tikzcd} \tag{5.2}$$

Then

$$p\varphi : E \to I, \qquad \begin{aligned} \widetilde{H} : E \times I &\to E \\ (e,t) &\mapsto H(e, \min\{t, p(\varphi(e))\}) \end{aligned} \tag{5.3}$$

defines a Strøm structure on $E_A \hookrightarrow E$. ∎

Note that 'cofibration' is meant here in the sense of *unpointed* cofibration. In the sequel we will work with pointed spaces. It is the task of the reader to understand how to use the above result in the pointed context.

Before moving on we really must comment on how quirky this theorem is. It has no dual. The pushout of a fibration will almost never be a fibration. The reader should take some time to think about this.

6 The Bott-Samelson Theorem

Throughout this section X will be a well-pointed, path-connected space. The reduced suspension of X is defined by the pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & C_+X \arrow[d] \\ C_-X \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \Sigma X \end{tikzcd} \tag{6.1}$$

where $C_\pm X$ are copies of the reduced cone. Since X is well-pointed the inclusions $X \hookrightarrow C_\pm X$ are both pointed and unpointed cofibrations, so (6.1) is a pushout and a homotopy pushout in both Top and Top_*. We will be working in Top_* throughout, but have been careful about our assumptions for a reason.

We begin by taking the path space fibration $e_0 : P\Sigma X \to \Sigma X$ and forming pullbacks over the spaces in (6.1) to obtain a strictly commutative cube

$$\begin{tikzcd} E_0 \arrow[rr] \arrow[dr] \arrow[dd] & & E_+ \arrow[dr] \arrow[dd] & \\ & E_- \arrow[rr, crossing over] & & P\Sigma X \arrow[dd, "e_0"] \\ X \arrow[rr] \arrow[dr] & & C_+X \arrow[dr] & \\ & C_-X \arrow[rr] \arrow[from=uu, crossing over] & & \Sigma X. \end{tikzcd} \tag{6.2}$$

Exercise 6.1 Show that the top face of (6.2) is both a pushout and a homotopy pushout.□

Now it is easy to identity the (fibre) homotopy types of the three space E_+, E_-, E_0. We will want to be a bit picky with exactly how we do so, however. Recall the suspension map $\sigma = \sigma_X : X \to \Sigma\Omega X$, which was defined in the previous section. We define $\nu : X \times \Omega\Sigma X \to \Omega\Sigma X$ to be the composite

$$X \times \Omega\Sigma X \xrightarrow{\sigma \times 1} \Omega\Sigma X \times \Omega\Sigma X \xrightarrow{\mu} \Omega\Sigma X \tag{6.3}$$

where μ is the loop multiplication.

Exercise 6.2 Show that there is a homotopy commutative diagram

$$\begin{tikzcd} \Omega\Sigma X \arrow[d, "\alpha_-"', "\simeq"] & X \times \Omega\Sigma X \arrow[l, "\nu"'] \arrow[r, "pr_2"] \arrow[d, "\alpha_0"', "\simeq"] & \Omega\Sigma X \arrow[d, "\alpha_+"', "\simeq"] \\ E_- & E_0 \arrow[l] \arrow[r] & E_+ \end{tikzcd} \tag{6.4}$$

in which each vertical arrow is a homotopy equivalence. (Hint: Work right to left.) □

Exercise 6.3 Use Exercise 6.2 to show that

$$\nu_* \oplus pr_2 : H_*(X \times \Omega\Sigma X) \to H_*\Omega\Sigma X \oplus H_*\Omega\Sigma X \tag{6.5}$$

is an isomorphism. Conclude that if H_*X is torsion free or if field coefficients are used, then

$$\nu_* : \widetilde{H}_*X \otimes H_*\Omega\Sigma X \to \widetilde{H}_*\Omega\Sigma X \tag{6.6}$$

is an isomorphism, where the tilde denotes reduced homology. □

Next we'll want to explore the algebraic implications of (6.6). Our conventions on grading are discussed in the appendix.

Lemma 6.1 *Fix a principal ideal domain*[1] *R. Let A be a connected associative graded algebra and M a graded module which is R-free in each degree and has $M_0 = 0$. Suppose there is a graded homomorphism $f : M \to \overline{A}$. Then the extension $\overline{f} : T(M) \to A$ is an algebra isomorphism if and only if the composition*

$$M \otimes A \xrightarrow{f\otimes 1} \overline{A} \otimes A \xrightarrow{\mu} \overline{A} \tag{6.7}$$

is an isomorphism, where μ denotes the algebra multiplication on A.

The tensor algebra $T(M)$ is discussed in the appendix. We recall graded means graded by the non-negative integers, that A is connected if $A_0 \cong R$, and that $\overline{A} = \bigoplus_{n\geq 0} A_n$.

Exercise 6.4 Prove Lemma 6. (Hint: One direction is easy. For the other use induction to show that (6.7) is an isomorphism in each degree). □

We now have all the pieces we need to assemble the important result which we would like to state this week.

Theorem 6.2 (Bott-Samelson) *Fix a principal ideal domain R. Assume that X is a (path-)connected well-pointed topological space such that H_*X is R-free. Then the map*

$$\sigma_* : \widetilde{H}_*X \to H_*\Omega\Sigma X \tag{6.8}$$

induces an algebra isomorphism

$$T(\widetilde{H}_*X) \xrightarrow{\cong} H_*\Omega\Sigma X. \tag{6.9}$$

Exercise 6.5 Prove the Bott-Samelson Theorem 6.2. □

[1] In our applications R will be either $\mathbb{Z}$ or one of the field $\mathbb{Z}_p$, p prime, $\mathbb{Q}$. You may assume that R is one of these if you prefer.

7 Loop Spaces of Spheres

A good application for the Bott-Samelson Theorem is to compute the homologies of loop spaces of spheres. Homology and cohomology in this section will be taken over the integers.

Exercise 7.1 Compute the Pontryagin algebra $H_*\Omega S^{n+1}$ with integral coefficients. $\Box$

On the other hand, to compute the cohomology rings of loop spaces of spheres we need to work a little bit more.

Exercise 7.2 Compute the cohomology ring $H^*\Omega S^{n+1}$. (Hint: First use Proposition 3.1 to compute the coproduct Δ_*. The structure of $H^*\Omega S^{n+1}$ depends on the parity of n. I will accept a computation with rational coefficients if you find the algebra difficult.) $\Box$

A Appendix: The Tensor Algebra

Let R be a commutative ring with unit. We will work with graded modules over R. Here graded will mean graded over the non-negative integers, so such an object M has a decomposition

$$M = \bigoplus_{n \geq 0} M_n \tag{A.1}$$

where each M_n is a R-module. In formulas we will understand $M_{-n} = 0$ whenever $n > 0$. If $x \in M_n$ is a homogeneous element, then we write $|x| = n$ to denote its **degree**. We will consider R to be a graded module concentrated in degree 0. We restrict to modules of **finite type**, which means that each M_n will be a finitely generated R-module. Moreover our modules will be **connected**, which means that an isomorphism $R \cong M_0$ is fixed. We write

$$\overline{M} = \bigoplus_{n \geq 0} M_n \tag{A.2}$$

for the positive part of M.

A **homorphism** $f : M \to N$ between graded modules M, N consists of a family of R-module homomorphisms $f_n : M_n \to N_n$. The **kernel** of a graded homomorphism $f : M \to N$ is the graded module $\ker(f)$ with $\ker(f)_n = \ker(f_n) \subseteq M_n$. The **cokernel** of f is the graded module $coker(f)$ with $coker(f)_n = coker(f_n)$.

If M, N are graded modules, then their **graded tensor product** is the graded module $M \otimes N$ with

$$(M \otimes N)_n = \bigoplus_{i+j=n} M_i \otimes_R N_j. \tag{A.3}$$

The reader can check that the graded tensor product is associative. Moreover $M \otimes R \cong M \cong R \otimes M$ for any graded module M.

Given graded modules M, N, the homomorphism $T : M \otimes N \to N \otimes M$ defined on homogeneous elements by

$$T(m \otimes n) = (-1)^{|m||n|} n \otimes m. \tag{A.4}$$

is an isomorphism called the **twist**, or **switching**, map. In general we use the Koszul sign rule throughout, so a graded minus sign is introduced whenever the order of two elements is interchanged.

If $f : M \to M'$ and $f' : N \to N'$ are homomorphisms, then $f \otimes f' : M \otimes N \to M' \otimes N'$ is the homomorphism with $(f \otimes f')_n = \bigoplus_{i+j=n} f_i \otimes f'_j$. If M is a graded module, then its **dual module** M^* is that with

$$(M^*)_n = Hom_R(M_n, R). \tag{A.5}$$

If $f : M \to N$ is a homomorphism, then $f^* : N^* \to M^*$ is the homomorphism with $(f^*)_n = Hom(f, 1)$.

Definition 1 *A graded **algebra** over R is a graded module A equipped with a pair of homomorphisms $\mu : A \otimes A \to A$ and $\eta : R \to A$ which make the following diagram commutes*

$$\begin{tikzcd}
A \otimes R \arrow[d, "1\otimes\eta"'] \arrow[dr, "\cong"] & \\
A \otimes A \arrow[r, "\mu"] & A \\
R \otimes A \arrow[u, "\eta\otimes 1"] \arrow[ur, "\cong"'] &
\end{tikzcd} \tag{A.6}$$

*The homomorphism μ is called the **product** of A and the homomorphism η is called its **unit**. The algebra A is said to be **associative** if the left-hand diagram below commutes, and **commutative** if the right-hand diagram below commutes*

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A.
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[rr, "T"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd} \tag{A.7}$$

where T is the twist map.

*If (A, μ_A, η_A) and (B, μ_B, η_B) are algebras, then a graded homomorphism $f : A \to B$ is said to be an **algebra map** if it makes both the following diagrams commute.*

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mu_A"] \arrow[d, "f\otimes f"'] & A \arrow[d, "f"] \\
B \otimes B \arrow[r, "\mu_B"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\
A \arrow[rr, "f"] & & B.
\end{tikzcd} \tag{A.8}$$

$\blacksquare$

Definition 2 *Let M be a graded module. The **tensor algebra** on M is the graded algebra $T(M)$ defined as follows. We set*

$$T(M) = \bigoplus_{n \geq 0} M^{\otimes n} \tag{A.9}$$

where we set $M^{\otimes 0} = R$ and inductively define $M^{\otimes n} = M \otimes M^{\otimes n-1}$ for $n \geq 1$. The algebra product is defined by concatenation. For $x_1 \otimes x_2 \otimes \ldots \otimes x_m \in M^{\otimes m}$ and $y_1 \otimes y_2 \otimes \ldots \otimes y_n \in M^{\otimes n}$ we put

$$(x_1 \otimes x_2 \otimes \ldots \otimes x_m) \cdot (y_1 \otimes y_2 \otimes \ldots \otimes y_n) \in M^{\otimes m+n} \tag{A.10}$$

and extend this in the obvious way to an R-linear product $\mu : T(M) \otimes T(M) \to T(M)$. The product is associative by construction but fails to be commutative.

There is a homomorphism of modules

$$\iota_M : M \to T(M) \tag{A.11}$$

which identifies $M = M^{\otimes 1}$. Clearly the image of ι_M generates $T(M)$ as an algebra. $\square$

Proposition A.1 *Let M be a graded module. Assume that $f : M \to A$ is a module homomorphism into a associative algebra A. Then there is a unique algebra homomorphism $\overline{f} : T(M) \to A$ making*

$$\begin{tikzcd} M \arrow[d, "\iota_M"'] \arrow[dr, "f"] & \\ T(M) \arrow[r, "\overline{f}"'] & A \end{tikzcd} \tag{A.12}$$

commute.

Proof For $x_1 \otimes x_2 \otimes \ldots \otimes x_n \in M^{\otimes n}$ put

$$\overline{f}(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = f(x_1) \cdot f(x_2) \ldots f(x_n). \tag{A.13}$$

This is well-defined since the associativity of A implies that the right-hand side makes sense regardless of any particular bracketing scheme. The definition (A.13) is R-linear, and extending it in the obvious way yields a homomorphism $\overline{f} : T(M) \to A$ with $\overline{f}\iota_M = f$. Clearly $\overline{f}$ is an algebra map. Since the image of ι_M generates $T(M)$ as an algebra, $\overline{f}$ is the unique algebra map satisfying $\overline{f}\iota_M = f$. ∎

Using the proposition we turn $T(-)$ into a functor. If $f : M \to N$ is a map of graded modules, then we let

$$T(f) : T(M) \to T(N) \tag{A.14}$$

be the unique algebra map induced by the composition $M \xrightarrow{f} N \xrightarrow{\iota_N} T(N)$. We can now view $T(-)$ as a functor from the category of graded modules to the category of associative graded algebras. The reader can interpret Proposition A.1 as the statement that $T(-)$ is right adjoint to the forgetful functor in the opposite direction.

References

[1] N. Steenrod, N., D. Epstein, *Cohomology Operations*, Ann. Math. **50**, Princeton Univ. Press, (1962).